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On quantum spin-chain spectra and the structure of Hecke algebras and q-groups at roots of unity

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Abstract. We combine statistical mechanical and algebraic Lie theory techniques to investigate the ordinary representation theory of the quotients $H_n^N(q)$ of the Hecke algebras $H_n(q)$ when q is a root of unity. We show how to determine the main invariants of these algebras (the standard module contents of the indecomposable projective modules) in general N. We give complete explicit results in the cases $N = 2, 3$.

These results are used to determine the q-variation energy level convergences in $U_q(sl(N))$ invariant quantum spin chains.

1. Introduction

The generically irreducible ('standard') representations of the Hecke algebra $H_n(q)$ [50] have well defined multiplicities as composition factors in the quantum spin-chain representation $V_N^{\otimes n}$ for any q. These multiplicities are given by the dimensions of certain irreducible representations, Weyl modules, of the classical enveloping algebra $U(sl_N)$ [12, 31]. Thus, in particular, these multiplicities are independent of q. This is a well known consequence of Schur–Weyl duality [61] and character properties of the standard representations. The connection between the reducibility of these standard representations of $H_n(q)$ at roots of unity and increased Hamiltonian spectrum degeneracy (i.e. energy level convergence) of $U_q(sl_N)$ invariant spin chains [54] is in turn a direct consequence of this result. It has not previously been possible to compute the increased degeneracies systematically, or to say which sectors of the spectrum are converging to produce them. Furthermore, it has not generally been possible to say which degeneracies observed on a given finite size chain would survive to the thermodynamic limit. Some specific data has been available for small n (see [36, 50, 21] and references therein). Lascoux *et al* [40, 7] produced an elegant scheme for obtaining the general data in principle, but requiring large amounts of computation in practice. In this paper we show how to compute the complete answer for any n and N in a relatively straightforward construction merging technology from statistical mechanics and modular representation theory (leaning heavily, in particular, on the results of Soergel [58], Lusztig [43], Jantzen [34] and Donkin [28]). We explain this technology in terms familiar in physics—specifically, Young's representation theory of the symmetric group [61, 31]. Our results give the true irreducible content of the generically irreducible modules, and dually give the (q-deformation of classical) Weyl module content of the indecomposable components of $V_N^{\otimes n}$ *as a* $U_q(sl_N)$ *module*. These indecomposable 'tilting' modules [28] are *larger* at roots of unity than in the classical case, and since the dimensions of classical Weyl modules are well known we can compute the larger dimensions of the tilting modules (i.e. the multiplicities of irreducibles in $V_N^{\otimes n}$ on the Hecke side) if we know their Weyl content.

The $U_q(sl_{N|M})$-symmetric n-site quantum spin-chain Hamiltonian [56, 54, 19] is

$$\mathcal{H}^{N,M} = \sum_{j=1}^{n-1} R_j^{N,M} \tag{1}$$

where

$$R_j^{N,M} = \frac{q+q^{-1}}{2} - \left\{ \sum_{a \neq b} E_j^{ab} E_{j+1}^{ba} + \frac{q+q^{-1}}{2} \sum_{a=1}^{M+N} \epsilon_a E_j^{aa} E_{j+1}^{aa} \right.$$
$$\left. + \frac{q-q^{-1}}{2} \sum_{a\neq b} \frac{a-b}{|a-b|} E_j^{aa} E_{j+1}^{bb} \right\}.$$

Here $E_j^{a,b}$ denotes the (a,b)th elementary matrix acting on the jth tensor factor of $(\mathbb{C}^{N+M})^{\otimes n}$, and $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_N = -\epsilon_{N+1} = -\epsilon_{N+2} = \cdots = -\epsilon_{N+M} = 1$. The main problem is to determine the large n limit Hamiltonian eigenvalue spectrum, the mass gap (as in [41]), and the spectrum degeneracies [52]. Several cases for the pair of integers (N, M) are known to be physically significant. For example, $(2, 0)$ is the spin-$\frac{1}{2}$ Heisenberg chain; $(1, 1)$ is thought to model metal surface absorption properties [37]: $(2, 1)$ is relevant for understanding Anderson's t–J model [5, 10, 30]. The same matrices appear in a certain class of asymmetric diffusion problems now thought to model aspects of traffic flow, interface growth, the dynamics of shocks and various other interesting phenomena (see [29, 3] and references therein). Closely related models are also relevant for a wide variety of many-body cooperative effects, such as critical phenomena, for computation in QCD, areas of quantum chemistry [55], nuclear physics [59], and string/conformal field theory [39, 46, 60]. These spin-chain models are also integrable, i.e. amenable, at least in principle, to the Bethe ansatz [1, 2], the construction of Yang–Baxter equations, and other methods of exact solution. Such models have naturally been the subject of intense study worldwide for several years [11, 24, 44, 18, 16].

Through the appearance of the Yang–Baxter equations we may extract a representation of a quotient of $H_n(q)$ from each model. The algebra $H_n(q)$ is given by generators $U_j (1 \leqslant j < n)$ and relations

$$\begin{aligned} &U_i U_i = (q+q^{-1})U_i \\ &U_i U_{i\pm1} U_i - U_i = U_{i\pm1} U_i U_{i\pm1} - U_{i\pm1} \\ &U_i U_{i+j} = U_{i+j} U_i \qquad (j \neq 1) \end{aligned} \tag{2}$$

and the model representation is defined by $U_j \mapsto R_j^{N,m}$. We denote by $H_n^{N,M}(q)$ the image of $H_n(q)$ under this representation. It is the representation theory of $H_n^{N,M}(q)$ which controls the Hamiltonian spectrum degeneracies in each case [51]. Relatively little is known of the general case at q a root of unity, but certain simplifications occur in the case $H_n^{N,0}(q)$, written $H_n^N(q)$, and it is this case we study here. (However, the general case remains an important problem, and our approach stays, as far as possible, with techniques applicable to the general case.)

The invariance situation for $M = 0$ is summarized by the following 'q-Schur–Weyl duality diagram', in which the action of the algebras shown on the left on $V_N^{\otimes n}$ commutes with the action of those shown on the right (the full lines are surjective algebra homomorphisms):

$$
\begin{tikzcd}
H_n(q) \arrow[dr, dashed] \arrow[dd] & & U_q sl_N \arrow[dl, dashed] \arrow[dd] \\
& V_N^{\otimes n} & \\
H_n^N(q) \arrow[ur, dashed] & & S_q(n, N) \arrow[ul, dashed]
\end{tikzcd}
\tag{3}
$$

Here $H_n^N(q) \cong \mathrm{End}_{U_q(sl_N)}(V_N^{\otimes n})$ [47] is the quotient of $H_n(q)$ which acts faithfully on $V_N^{\otimes n}$, and $S_q(n, N)$ is the *q-Schur algebra* [20], the quotient of the q-group which acts faithfully on $V_N^{\otimes n}$.

We do not concern ourselves here with the characterization of the $U_q(sl_N)$ invariant spin-chain spectrum by momentum (cf [54, 1]), or with the numerical values of Hamiltonian eigenvalues. Thus, in terms of the concrete applications discussed above, this work is far from the end of the story (cf [41, 52]). However, we are able, efficiently and elegantly, to encode a significant amount of useful level crossing data. In order to *extract* this data we must apply some fairly technical mathematics. The good news is that, once extracted, the data may be *presented* in a simple way, as we will see. The technical effort is worthwhile, since this data *must* be controlled before the spin-chain spectra can be analysed in a physically useful way (cf [17, 51]). The computations for the results we present also have the interesting appearance of crystal growth in various dimensions, which itself may prove useful in asymmetric diffusion problems (cf [29]; this aspect will be discussed elsewhere).

For q a primitive lth root of unity (care is needed if l is not an odd prime greater than N) the representation theory of $H_n(q)$ is greatly altered from the classical or generic situation, in which $H_n(q) \cong \mathbb{C}S_n$, the symmetric group algebra, and $U_q(sl_N) \cong U(sl_N)$. However, the index set for labelling isomorphism classes of irreducible representations is basically unchanged. We wish to import some standard results from algebraic Lie theory, so we begin in section 2 by recalling the relevant features of the classical index set in weight lattice formalism.

In sections 3 and 4 we motivate the introduction of some modern algebraic Lie theory technology, and illustrate the nature and physical interpretation of our results by giving the $N = 2, 3$ cases in some detail. Specifically, for each indecomposable projective module P'_λ and standard module Δ'_μ we give $D_{\lambda\mu}$, the number of times Δ'_μ appears as a composition factor in P'_λ. We show how all the other multiplicities we have discussed may be computed from these. By adopting notions of *quasi-heredity* [14, 26] and *tilting modules* [28] we can reduce the computation of Hamiltonian degeneracies to a simply stated (although richly structured) algorithmic procedure. In section 5 we explain this procedure in general.

In this paper we use the definitions in [50] on the Hecke algebra side, and those of Chari and Pressley [13] on the q-group side, with some minor additions mentioned in the next section.

2. Index sets for irreducible representations

The theory behind root systems and weight lattices for Lie algebras, and its connection with the representation theory of $\mathbb{C}S_N$ and other Coxeter groups, is well documented, e.g. in O'Raifeartaigh [53]. We only review here the components we use most intensively.

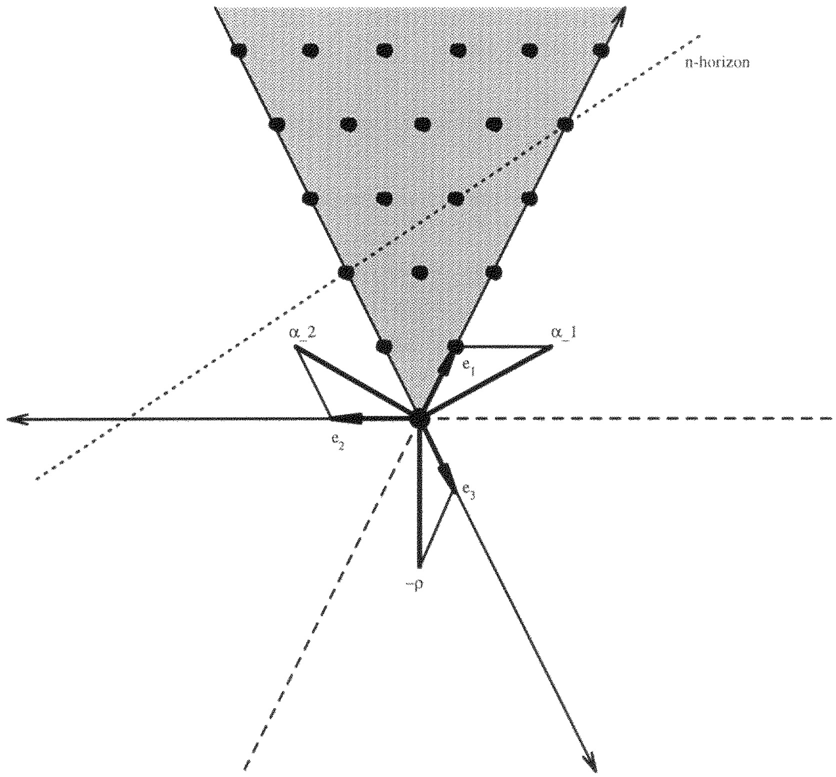

Figure 1. Sketch of the root/weight space for A_2 (i.e. $\mathbb{R}^3$ viewed from the e_0 direction—note that in this case $w_1 \sim e_1, w_2 \sim e_1 + e_2$). The dots are the dominant weights. The n-horizon is shown for the case $n = 4$. For example, the three weights on the 4-horizon are, from left to right, $\lambda = (0, 2), (2, 1), (4, 0)$, giving integer partitions $(2, 2), (3, 1), (4)$ respectively, while $\lambda = (1, 0)$ corresponds at $n = 4$ to partition $(1) + (1, 1, 1) = (2, 1, 1)$.

2.1. The lattice of weights and the Weyl groups

Fix N and let $V = V_N = \mathbb{R}^N$, with $\{e_i | i = 1, 2, \ldots, N\}$ an orthonormal bias with respect to the standard inner product $(,)$. Put $e_0 := \frac{1}{N}\sum_{i=1}^N e_i$. Recall that sl_N is associated to the A_{N-1} Coxeter system [33] which involves Weyl groups of reflections of the space V which fix the line $\mathbb{R}e_0$ or equivalently the hyperplane of $\mathbb{R}^N$ perpendicular to e_0. The primitive or simple roots (in Okubo formalism) are $\Pi = \{\alpha_i = e_i - e_{i+1} | i = 1, \ldots, N-1\}$, the highest root is $\tilde{\alpha} = e_1 - e_N$, the roots are $\Phi = \{e_i - e_j | i \neq j\}$, positive roots $\Phi^+ = \{e_i - e_j | i < j\}$, and the primitive (or fundamental dominant) weights are $\{\omega_j = \sum_{i=1}^j (e_i - e_0) | j = 1, 2, \ldots, N-1\}$.

In this framework the Weyl group $W = S_N$ acts by permuting $e_i \mapsto e_{\pi(i)}$ (hence it is the copy of S_N generated by the reflections reflecting primitive roots: $\sigma_i : e_i \mapsto \pm e_i$). The *weight lattice* X is the $\mathbb{Z}$-span of the primitive weights. It will suffice for our present purposes to illustrate this with the picture for A_2 (figure 1). Note that the shaded ‘dominant’ region in the figure is a fundamental domain for the Weyl group action.

The weight lattice X is preserved by the Weyl group action, and the intersection X^+ of X with the shaded fundamental domain is thus a set of representative elements from each of the Weyl group orbits of X. Thus X^+ indexes possible highest weights, and hence simple modules of $U(sl_N)$.

Note that $\lambda \in X^+$ is of the form

$$\lambda = \sum_{i=1}^{N-1} \lambda_i \omega_i \qquad \lambda_i \in \mathbb{N}_0$$

so that dominant weights may be regarded as integer partitions via

$$\lambda \mapsto (\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1}, \lambda_2 + \cdots + \lambda_{N-1}, \ldots, \lambda_{N-1}).$$

For $n \in \mathbb{N}$ we define the n-horizon as the hyperplane $\sum_i i\lambda_i = n$, and X^n as the set of dominant weights below and on the n-horizon whose degree is congruent to n modulo N. The usual index set for the simple modules of $\mathbb{C}S_n$, is the set of integer partitions [61] of degree n. For the quotient in the classical Schur–Weyl duality

$$0 \mapsto \mathbb{C}S_n Y_N^1 S_n \to \mathbb{C}S_n \to \mathrm{End}_{U(sl_N)}(V^{\otimes n}) \to 0 \tag{4}$$

(a short exact sequence with Y_N^1 the level N Young symmetrizer [31, 61]) the corresponding index set is the set of integer partitions of degree n and up to N parts. This set is obtained from the set X^n by adding to each weight a multiple of the N-vector $(1, 1, 1, \ldots, 1)$ such that the resulting degree is n (see figure 1 for an example). We shall not distinguish between X^n and its image under this map hereafter.

We partially order X by $\mu \trianglerighteq \lambda$ if $\mu - \lambda$ can be expressed as a linear combination of simple roots with positive coefficients (note that this *dominance order* coincides, where applicable, with the usual order on partitions of n).

2.2. The l-affine Weyl group

Recall that there is an affine Weyl group action on V defined as follows (again we precis Humphreys [33], but take advantage of certain simplifications in the A_{N-1} case). For α a root, $k \in \mathbb{Z}$, define the affine hyperplane

$$H_{\alpha,k} = \{\lambda \in V | (\lambda + \rho, \alpha) = k\} \qquad \left(\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha\right)$$

and hence an affine reflection

$$s_{\alpha,k}(\lambda) = \lambda - ((\lambda + \rho, \alpha) - k)\alpha.$$

(Note that in this 'ρ-shifted' set-up the $k = 0$ reflections generate a copy of W which fixes the point $-\rho$ rather than 0.) for $l \in \mathbb{N}$ put

$$\mathcal{H}_l = \{H_{\alpha,kl} | \alpha \in \Phi^+, k \in \mathbb{Z}\}.$$

Fixing $l \in \mathbb{N}$ the l-affine Weyl group may be realized as

$$W_l = \langle s_{\alpha,kl} | \alpha \in \Phi^+, k \in \mathbb{Z}\rangle.$$

Define $\mathcal{A}$ as the set of 'alcoves' of V—the set of the connected components of $V \setminus \cup_{H \in \mathcal{H}l} H$. In particular, the 'fundamental' alcove is

$$A^0 = \{\lambda \in V | 0 < (\lambda + \rho, \alpha) < l \,\forall \alpha \in \Phi^+\}$$

with bounding hyperplanes $\{H_{\alpha,0}, \alpha \in \Pi\} \cup \{H_{\tilde{\alpha},l}\}$. A set of generators of W_l is

$$\mathbb{S}_l = \{s_{\alpha,0}, \alpha \in \Pi\} \cup \{s_{\tilde{\alpha},l}\}.$$

Note that W_l permutes the set $\mathcal{A}$. This characterization of the l-affine Weyl group greatly facilitates a description of the representation theory of $U_q(sl_N)$ and $H_n^N(q)$ at q an lth root

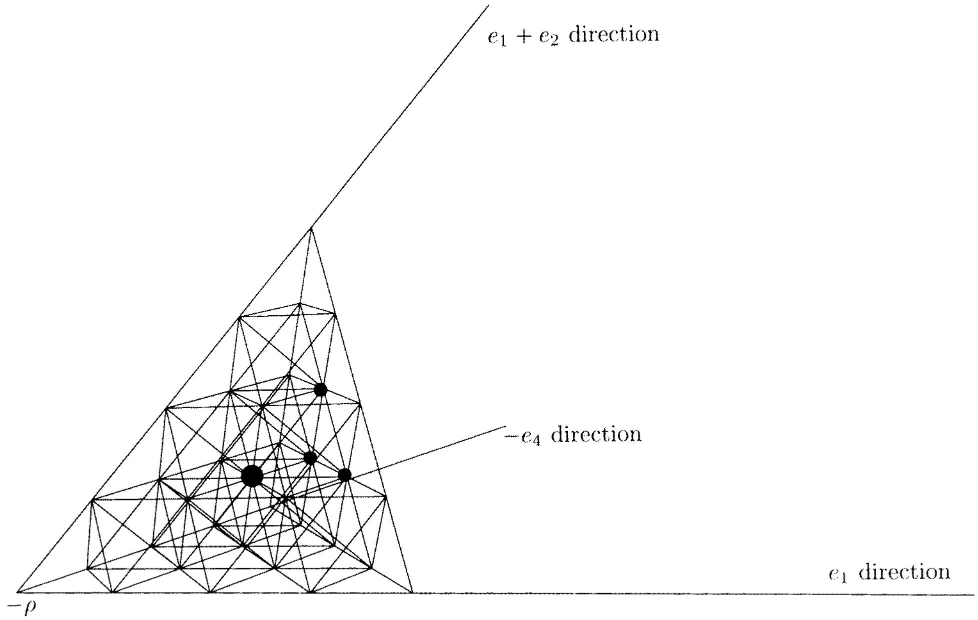

Figure 2. Sketch indicating the fundamental alcove A^0 and the dominant region for the Coxeter system A_3 after ρ shifting in the case $l = 4$. The large dot is $(0, 0, 0)$, the unique dominant weight in the interior of the fundamental alcove in this case. The small dots are the fundamental dominant weights $\omega_1 = (1, 0, 0), \omega_2 = (1, 1, 0), \omega_3 = (1, 1, 1)$—the dominant weights on the first affine hyperplane in this case.

of unity, as we shall see. From now on by the affine Weyl group we mean the l-affine Weyl group.

Note that each connected component of $V \setminus \cup_{H \in \mathcal{H}_0} H$ is a fundamental domain of the (ρ-shifted) ordinary Weyl group action. The *dominant region* of V is now defined as the fundamental domain which contains the fundamental alcove A^0. Write $A^+ \subset \mathcal{A}$ for the set of alcoves in the dominant region.

For example consider A_3 at $l = 4$. Here

$$X \cap A^0 = \left\{ \sum_i a_i \omega_i | a_i \in \mathbb{Z}, 0 < (\lambda + \rho, \alpha) < 4 \, \forall \alpha \in \Phi^+ \right\}$$

so (a_1, a_2, a_3) satisfies $-1 < a_i < 3$ and $-3 < (\sum_i a_i) < 1$ giving $(a_1, a_2, a_3) = (0, 0, 0)$ as the only element. A sketch indicating the closure of A^0 in this case appears in figure 2—from which it can be seen that $(0, 0, 0)$ is indeed the only interior lattice point.

The hyperplanes may be thought of as making an (undirected) simplicial complex of V, with the alcoves the codimension 0 open simplices. The general open simplices are called facets. Write $\mathcal{A}_i$ for the set of codimension i facets. Thus $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_1$ is the set of alcove 'walls', each a connected component of $M \setminus \cup_{H \in \mathcal{H}_l \setminus \{M\}} H$ for some hyperplane M. For $\lambda \in X$ we write F_λ for the unique facet containing λ. We write $\mathrm{St}(\lambda)$ for the group $\{w \in W_l | w \cdot \lambda = \lambda\}$. For any facet F let $\bar{F}$ denote its closure. Thus $\overline{A^0}$ is a fundamental domain for the action of the affine Weyl group. We put $C = \overline{A^0} \cap X$.

We define a length function on the dominant region of V by

$$\mathrm{len}(x) = \#\text{ hyperplanes between } x \text{ and } 0.$$

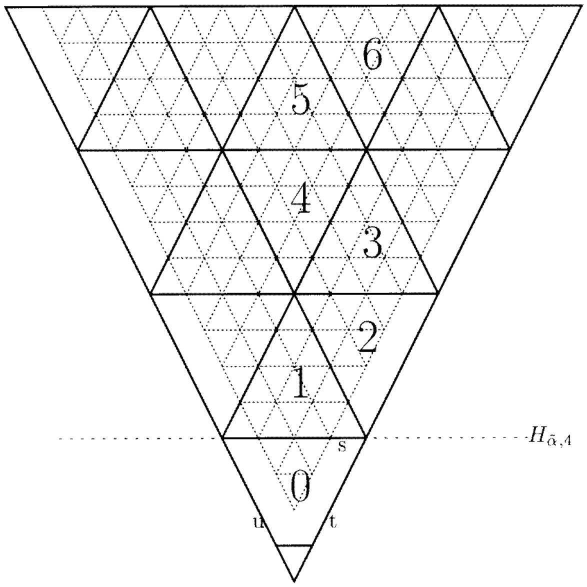

Figure 3. Walls and alcoves in the dominant region of A_2 in the case $l = 4$. Writing $\{s, t, u\}$ for $\mathbb{S}_l$ as indicated (e.g. s is a reflection in $H_{\tilde{\alpha},4}$) we have marked the lengths of the alcoves on a Bruhat increasing path with $0 = \mathrm{len}(A^0)$; $1 = \mathrm{len}(sA^0)$; $2 = \mathrm{len}(stA^0)$; $3 = \mathrm{len}(stuA^0)$; $4 = \mathrm{len}(stutA^0)$; $5 = \mathrm{len}(stutsA^0)$; and $6 = \mathrm{len}(stutstA^0)$.

Note that this function is well defined on the set of alcoves $\mathcal{A}$. Note also that in the case $l = 1$ this length orders X^+ by the dominance order. We define a 'Bruhat' order on $\mathcal{A}$ by $A \geqslant B$ if we can get from B to A by successive reflections which at each stage increase in length by 1. An example is shown in figure 3.

Note that any $A \in \mathcal{A}$ can be expressed as wA^0 for some $w \in W_l$. We hence define a 'right action' of the generators $\mathbb{S}_l$ of W_l on $\mathcal{A}$ by $As = wsA^0$. For example, consider figure 3 again. Note that As is the reflection of A in the unique wall of A (i.e. in the hyperplane touching A) which is in the (l)-affine Weyl orbit of the hyperplane corresponding to s.

Put

$$w_s(A) = (\overline{As} \cap \bar{A}) \cap \left(\bigcup_{S \in \mathcal{A}_1} S \right)$$

and let $\mathcal{A}_1^+$ be the set of alcove walls in the dominant region. Partition $\mathcal{A}_1^+$ by the W_l action (i.e. into $N + 1$ parts) and write $[t]$ for the class of wall t.

Note that there is a natural 1 : 1 correspondence between X^+ and the subset of $\mathcal{A}^+$ consisting of translations of A^0.

3. Representation theory preliminaries

3.1. Canonical representation types and invariants

For A an algebra, $M \in A$-mod (i.e. an element of the category of finite-dimensional left A-module) and $\mathcal{S}$ a set of inequivalent A-modules we put $M \in \mathcal{F}(\mathcal{S})$, and say M has an $\mathcal{S}$-filtration, if M, may be filtered by a finite series of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{m+1} = 0$$

such that for all $i = 0, \ldots, m$ the ith section $M_i/M_{i+1} \cong N_i \in \mathcal{S}$. If $|\{i|N_i = N\}|$ is independent of the choice of series for all M, N we write $(M : N)_{\mathcal{S}}$ for this filtration multiplicity (cf [50, equation (1.15)]). For example, let $\mathcal{L}$ be a complete set of (isomorphism classes of) simple modules of A. Then every M is in $\mathcal{F}(\mathcal{L})$ and we write $(M : N)_{\mathcal{L}}$ as $[M : N]$. Note that in each *simple* series M_1 is a maximal proper submodule of M. The intersection of all maximal proper submodules of M is $\operatorname{Rad} M$, and $\operatorname{Head} M = M/\operatorname{Rad} M$ is semisimple. Thus

$$M = M_0 \supset \operatorname{Rad} M \supset \operatorname{Rad}\operatorname{Rad} M \supset \cdots$$

is a finite series for M with semisimple sections.

Let Λ be an index set for $\mathcal{L}$. The smallest module with head $L_\lambda \in \mathcal{L}$ is obviously L_λ itself. There is also a unique largest with this property—the *projective* module denoted P_λ. We define a relation $\sim$ on Λ by $\lambda \sim \mu$ if $[P_\lambda : L_\mu] \neq 0$. The extension of this relation to an equivalence relation partitions $\mathcal{L}$ into *blocks*.

Hereafter we will write L_λ, P_λ for the simple and projective module of $S_q(n, N)$ with index $\lambda \in X^n$, and L'_λ, P'_λ for those of $H_n^N(q)$. For us, the crucial invariant of either algebra is its *Cartan matrix* $C_{\lambda\mu} := [P_\lambda : L_\mu]$.

For certain algebras (including quasi-hereditary algebras such as $S_q(n, N)$ and $H_n^N(q)$) we also have an intermediate set Δ (resp. Δ') of *standard* modules. A construction for the $H_n^N(q)$ standard modules was given in [50], and we give a definition in the appendix, but roughly speaking the standard module Δ_λ lies between L_λ and P_λ in size, again with head L_λ. These modules are such that the data $(P_\mu : \Delta_\lambda)_\Delta$, and similarly

$$D_{\mu\lambda} := (P'_\mu : \Delta'_\lambda)_{\Delta'} \tag{5}$$

are well defined, and such (in the cases we consider) that the data $(P_\mu : \Delta_\lambda)_\Delta$ are in 'reciprocity' with (i.e. numerically coincident with) the data $[\Delta_\lambda : L_\mu]$ [28]. Thus introducing standard modules 'halves' the difficulty of the calculation of $C_{\lambda\mu}$. For example, in the $H_n^N(q)$ case we have $C_{\lambda\mu} = \sum_\rho D_{\lambda\rho} D_{\mu\rho} = (DD')_{\lambda\mu}$. We will give examples in section 3.6. The tilting modules T_λ, T'_λ (the largest indecomposable summands of $V_N^{\otimes n}$) are for us a computational convenience essentially similar to the projectives.

Note that from our introduction, $(V_N^{\otimes n}, \Delta'_\lambda)_{\Delta'} = \dim(\Delta_\lambda)$, so the spectrum multiplicities of Hamiltonian $\mathcal{H}^{N,0}$ are regulated by $D_{\lambda\mu}$ for $H_n^N(q)$ via

$$[V_N^{\otimes n} : L'_\mu] = \sum_\lambda \dim(\Delta_\lambda) D_{\lambda\mu}.$$

We give a technical summary of quasi-heredity in appendix A. There we define certain sets of canonical indecomposable modules for $H_n^N(q)$ and $S_q(n, N)$ as outlined in the table below, each with the same index set $\{\lambda \in X^n\}$. The table is here to provide the physically motivated reader with the option of *skipping* appendix A.

We have

$$(T_\lambda : \nabla_\mu) = (P'_\lambda : \Delta'_\mu) = (\nabla'_\mu : L'_\lambda) \tag{6}$$

where the last equality is Brauer–Humphries reciprocity [28].

Fix l as before and put $U = U_q(sl_N)$. Let $\lambda \in C$ and define $\mathcal{M}_\lambda$ as the category of finite-dimensional U-modules M such that $(M : L_\mu)_L \neq 0$ only if $\mu \in W_l \cdot \lambda$. By the *linkage principle* [34] every $M \in U$-mod is uniquely expressible as

$$M = \bigoplus_{\lambda \in C} M^\lambda \tag{7}$$

with $M^\lambda \in \mathcal{M}_\lambda$. By the duality of table 1 (see also equation (33)) an exactly corresponding decomposition applies to $H_n^N(q)$-modules. In particular the composition matrices D block

Table 1. This table indicates how the canonical modules of $H_n^N(q)$ and $S_q(n, N)$ are related by the functors of q-Schur–Weyl/Ringel duality [22, 28] (denoted F, F', and defined explicitly in appendix A; see equation (33)).

$$
\begin{tikzcd}
 & V_N^{\otimes n} & \\
H_n^N(q) \arrow[ur, dashed] \arrow[rr, shift right, "F'"'] & & S_q(n, N) \arrow[ul, dashed] \arrow[ll, shift right, "F"']
\end{tikzcd}
$$

Simple modules:	L'_λ	L_λ
Standards:	Δ'_λ	Δ_λ
Costandards:	∇'_λ	∇_λ
Indecomp. Tilting:	T'_λ	T_λ
Indec. Projectives:	P'_λ	P_λ

diagonalize, with blocks $D(\lambda)$ labelled by $\lambda \in C$ and containing the rows and columns indexed by dominant weights in the W_l orbit of λ. For large enough n we will see that, amongst those $\lambda \in A^0$, $D(\lambda)$ does not actually depend on λ. Indeed, we have $D(\lambda) = D(\mu)$ if $F_\lambda = F_\mu$.

In these terms we may now give a preview of results for D.

3.2. Preview of results

We will take the example $N = 3$. Here it is possible to represent the whole of $D(\lambda)$, for λ on a particular type of facet, by a single picture. For $\lambda \in A^0$ we have figure 4, and for λ on any wall we have figure 5 (every λ on the *intersection* of walls gives a singleton block in this case, so with these we are done).

Consider the $\lambda \in A^0$ picture. Starting with the A^2 alcove diagram (the full lines in figure 3) we draw in each alcove wA^0 a shape indicating the location of the non-zero multiplicities $(P'_{w\lambda} : \Delta'_{w'\lambda})$ (in this case all multiplicities are either 0 or 1). The shape, or 'pattern', is simply another picture of the relevant part of the alcove diagram itself, scaled down to fit into the defining alcove wA^0, with alcove $w'A^0$ shaded if $(P'_{w\lambda} : \Delta'_{w'\lambda}) = 1$. We see that the typical arrangement is either a star or hexagon of alcoves. In every pattern the unique (Bruhat) highest among these shaded alcoves is wA^0 itself. In figure 5 the patterns are, correspondingly, of walls, and the pattern associated to each wall has been drawn *adjacent* to that wall. For example, the pattern associated to the s-wall of $stuA^0$ (i.e. the wall between $stuA^0$ and $stusA^0$; see also figure 3) shades that wall and also the s-wall of A^0 itself, but no others.

The proof of these results will be given shortly.

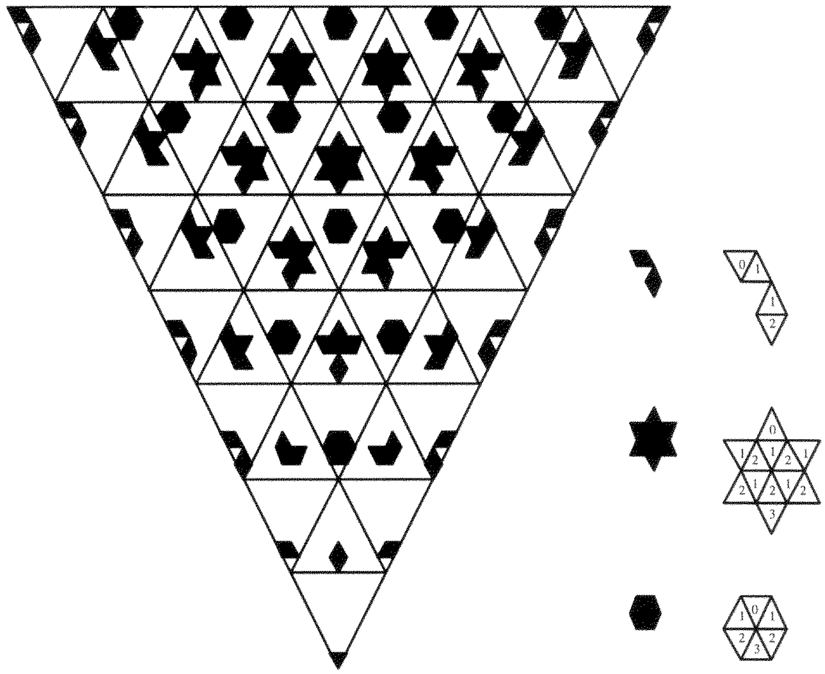

Figure 4. Schematic determining $D(\lambda)$ for $\lambda \in A^0$ in the case $N = 3$ (i.e. A^2). Patterns higher up the dominant region than shown follow the established pattern. For example, the pattern for *every* downward pointing alcove not touching the outside edge of the diagram is a hexagon. The numbers in the key set of patterns on the right will be explained later.

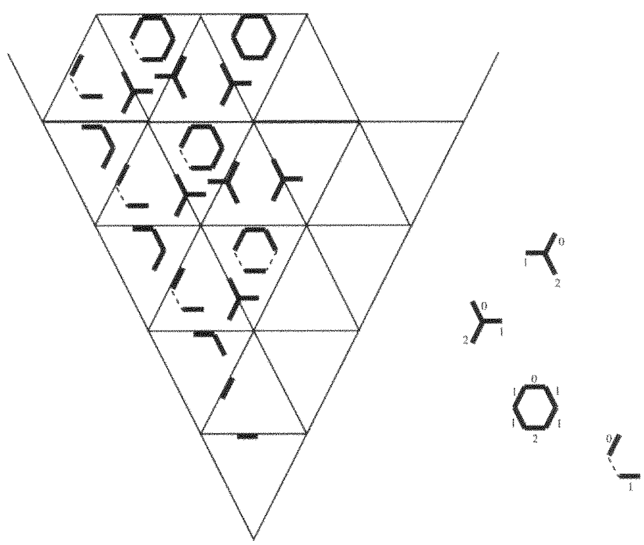

Figure 5. Schematic determining $D(\lambda)$ for each λ on a wall, i.e. wall patterns, for $N = 3$. Patterns not given in the figure (on the right in the alcove diagram) are obtained by left/right mirror symmetry. Patterns higher up the diagram than shown follow the established pattern. The key set of patterns on the right will be explained later.

3.3. *Interpretation of results*

A striking feature of the $N = 3$ results is that there are 'generic section patterns', each with only a small number of sections, close to the defining (top) section. By our general arguments the limit of the multiplicity of the simple module L'_λ in $V_N^{\otimes n}$ as $n \to \infty$ (and hence the multiplicity of each corresponding thermodynamic limit Hamiltonian eigenvalue) is given by

$$\dim(T_\lambda) = \sum_\mu D_{\lambda\mu} \dim(\Delta_\mu).$$

Our result shows that the sum is finite (and of course $\dim(\Delta_\mu)$ is always finite), so the corresponding multiplicities are all finite. Of course this is also true in the case $N = 2$, but there is no reason why it should be true in general. Indeed it can be shown by these methods that some multiplicities in the $N = 4$ case are infinite (i.e. there are arbitrarily complicated section patterns—with no limit on the number of sections involved—we will discuss this further elsewhere). Note that the situation for $(N, M) = (2, 1)$ is not yet clear, but is obviously of interest, in the light of our introductory discussion.

Note that it is an elementary exercise to verify from our results and equation (6) the cases of $[\nabla_\lambda : L_\mu]$ worked out by Doty and Sullivan [23].

It is also useful to consider the data in the form of $[\Delta'_\lambda : L'_\mu]$ (i.e. D'). This tells us which eigenvalues of the $M = 0$ Hamiltonian have level crossings with eigenvalues from the sector of the spectrum labelled by λ, as q varies through the lth root of unity. For example, in [50, p 5493] some specific explicit calculations determined the simple content of the $n = 8$ Specht module $S^{(4,3,1)}$ in the case $l = 4$. We verify this as follows. First $(4, 3, 1)$ goes to $(3, 2)$ in X^n, so we are looking at $\Delta'_{(3,2)}$. This lives in the alcove sA^0 (labelled 1 in figure 3). Consider some alcove B (say), and the simple module in the W_4 orbit of $(3, 2)$ which lives in alcove B. This appears in $\Delta'_{(3,2)}$ if the alcove labelled 1 in figure 3 appears in the pattern associated to alcove B in figure 4. In other words, thinking of the patterns as tiles to be placed (full sized) on the alcove diagram, L'_μ appears in $\Delta'_{(3,2)}$ if the pattern for μ overlays a region including alcove 1. Reading off from figure 4, the tiles which do this are those shown on the right in figure 6. The corresponding weights are

$$(3, 2), (5, 0), (8, 0), (7, 1), (4, 4), (12, 2), (9, 5), (8, 6), (11, 6), (9, 8), (11, 9).$$

Thus, there will be level crossings between eigenvalues in the $(3, 2)$ sector and all the other sectors shown in this list for large $n \equiv 5 (\mathrm{mod}\, 3)$, and in the thermodynamic limit. In order to recover specific finite-size data from this we simply use the 'localization' functor of [50] (cf [48]). Those weights above not killed by localization specifically to $n = 8$ are those on or below the 8-horizon, i.e.

$$(3, 2) \sim (4, 3, 1), (5, 0) \sim (6, 1, 1), (8, 0), (7, 1), (4, 4)$$

and thus we recover the result in [50] (and, adjusting n, all other finite results besides). Similarly from the left picture in figure 6, the only simples converging with the sector containing the normal ground state, $\lambda = (0)$ [51], are

$$(0), (4, 2), (9), (9, 9), (12, 6).$$

(This is true even in the limit of large n.)

We now turn to the general statement and proof of these results.

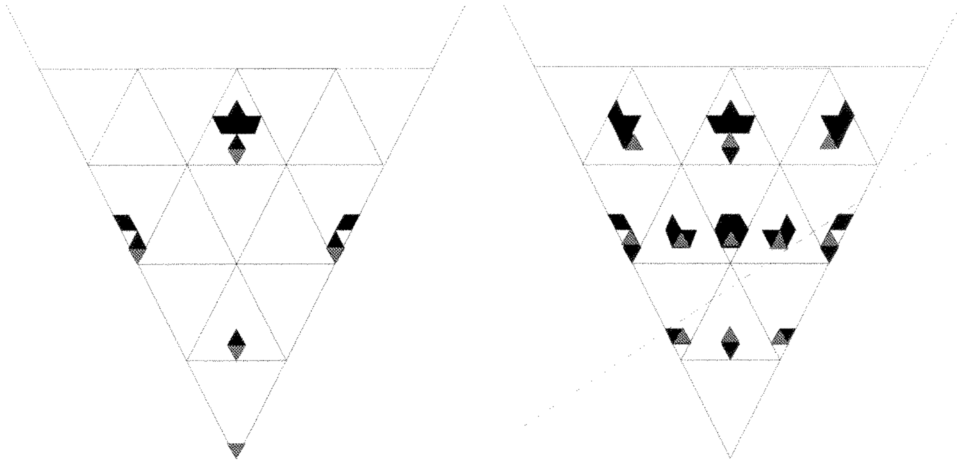

Figure 6. Alcove patterns which include the alcove $A = A^0$ (left picture) and $A = sA^0$ (right picture). The location of the alcove A on each pattern is indicated in grey. The broken line is an n-horizon.

3.4. Summary of Hecke algebra theory

Recall from [50] that the q-symmetrizer Y_N may be used to construct functors which embed H_n^N-mod in H_{n+N}^N-mod (via the algebra isomorphism

$$Y_N H_{n+N}^N Y_N \cong H_n^N$$

which is valid for $q[N]! \neq 0$—a restriction we henceforward assume to be satisfied). This embeds the tower

$$\cdots \subset H_n^N \subset H_{n+N}^N \subset H_{n+2N}^N \subset \cdots$$

in a *large* n limit algebra. Note that there is a tower for each conjugacy class of $n \bmod N$, and the union of their respective limits is called H^N. This 'global' algebra (cf [48]) has simple modules indexed by X^+ (as the natural limit of the $X^n s$). In the limit both induction and restriction via $H_n^N \subset H_{n+1}^N$ become functors on H^N-mod to itself, which we will denote Ind and Res, respectively. For λ a weight let $[\lambda] \in C$ denote the representative of the affine Weyl orbit of λ in C. Let Pr_λ denote the projection to $M^{[\lambda]}$ in the Hecke version of equation (7). Define composite functors $I_\lambda^\mu := \mathrm{Pr}_\mu \,\mathrm{Ind}\, \mathrm{Pr}_\lambda$ and $R_\lambda^\mu := \mathrm{Pr}_\mu \,\mathrm{Res}\, \mathrm{Pr}_\lambda$.

From [50] we find these functors preserve the property of standard filtrations. In particular the usual induction/restriction rules for symmetric groups [31] tell us that for each section Δ'_μ in $M \in \mathcal{F}(\Delta')$ we have sections

$$\Delta'_{\mu+e_1}, \Delta'_{\mu+e_2}, \ldots, \Delta'_{\mu+e_N} \tag{8}$$

in Ind M, where $\mu + e_i$ corresponds to adding a box to the ith row of the Young diagram of the weight μ, and the list runs over all cases where this produces another dominant weight (cf figure 1; there, starting from $\mu = (0, 0)$, $\mu + e_i$ is only dominant for $i = 1$; while for $\mu = (2, 1)$, say, $\mu + e_i$ is dominant for each of $i = 1, 2, 3$). The restriction Res M has sections $\Delta'_{\mu-e_i}$ similarly. We can think of the geometric structure on the set of (dominant) weights as being induced from these rules.

These functors also preserve projectivity. Thus, noting from [50] that $P'_0 = \Delta'_0$, we can gain information on the structure of projectives by working inductively on the dominance order. In particular, for an illustration, let us verify in the case $N = 3$ that

$$(P'_\lambda : \Delta'_\lambda)_{\Delta'} = 1 \qquad \text{and } (P'_\lambda : \Delta'_\mu)_{\Delta'} \neq 0 \text{ implies } \mu = \lambda \text{ or } \mu \trianglelefteq \lambda. \tag{9}$$

Suppose we apply Ind to some P'_λ for which, and for $P'_{\lambda'}$ below which, this is true. Then the image is a sum of indecomposable projectives including $P'_{\lambda+e_1}$ in particular (and note $\lambda + e_1 > \lambda$). Indeed $i < j$ implies $\lambda + e_i > \lambda + e_j$, so the big sum in

$$I_\lambda^{\lambda+e_1} P'_\lambda = P'_{\lambda+e_1} \oplus \bigoplus_\mu P'_\mu \tag{10}$$

can only contain projectives which are lower in order than $\lambda + e_1$. Every weight may be reached in this way, or by restriction, for which a similar argument applies.

3.5. Characters

Let us define a function χ which assigns to each standard filtered module M a list of natural numbers, one for each dominant weight μ, given by $(M : \Delta'_\mu)_{\Delta'}$. For example, $\chi(\Delta'_\lambda)$ is a list of almost all zeros, but with a 1 in the λ position. We call $\chi(M)$ the (*standard) character* of M. The character of a projective is called a projective character (note that $\chi(M)$ projective does not imply M projective).

Since we know the Δ'-content of Ind P'_λ from equation (8) the only problem in determining the content of $P'_{\lambda+e_1}$ is to determine, in cases where the Δ'-content of Ind P'_λ *contains* the Δ'-content of Ind P'_μ (some μ), whether this is just part of the content of $P'_{\lambda+e_1}$, or in fact a separate P'_μ summand.

For example, consider a dominant weight ν lying on a dimension 0 facet. The Jantzen sum formula [34] together with duality (or the Nakayama conjecture [35]) tells us that $P'_\nu = \Delta'_\nu$. Applying Ind once we learn that for all μ

$$(P'_{\nu+e_1} : \Delta'_\mu) \leqslant \sum_{i=1}^{N} (\Delta'_{\nu+e_i} : \Delta'_\mu) \tag{11}$$

i.e.

$$\chi(P'_{\nu+e_1}) \leqslant \sum_{i=1}^{N} \chi(\Delta'_{\nu+e_i}).$$

We will call an upper bound on the content of $P_{\nu+e_1}$ obtained in this way an *envelope* of $P_{\nu+e_1}$. Applying Ind again we naively have

$$\chi(P'_{\nu+e_1+e_1}) \leqslant \sum_{i,j=1}^{N} \chi(\Delta'_{\nu+e_i+e_j}). \tag{12}$$

However, all but N of the summands are in a different affine Weyl orbit to $\nu + 2e_1$, so (for l large enough) we deduce

$$\chi(P'_{\nu+2e_1}) \leqslant \sum_{i=1}^{N} \chi(\Delta'_{\nu+2e_i}). \tag{13}$$

Inducing again from this, and suitably projecting, we reach

$$\chi(P'_{\nu+2e_1+e_2}) \leqslant \sum_{\substack{i,j=1 \\ (i\neq j)}}^{N} \chi(\Delta'_{\nu+2e_i+e_j}). \tag{14}$$

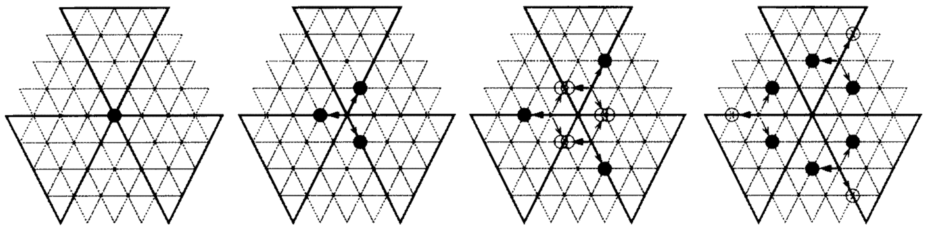

Figure 7. Moving from left to right, the first step is induction from a dimension zero facet to a wall; then translation along the wall by induction and projection; then translation off the wall into an alcove by induction and projection. The case illustrated is $N = 3, l = 4$.

where $\nu + 2e_1 + e_2$ lies on a dimension two facet. This process iterates in an obvious way until, at the Nth iteration, we deduce an envelope for a projective associated to a weight in an alcove which is a translation of A^0. Note that this envelope contains $N!$ sections (one in each of $N!$ alcoves). The case $N = 3$ is illustrated in figure 7.

In fact, as we will see shortly, this envelope gives the content of the corresponding projective exactly (and hence all those deduced before it are exact—and thus, indeed, for $N = 3$ they are among the patterns appearing in figures 4 and 5). This is a good illustration of the utility of the technique; however, it is not enough to determine D. In general, what we need is a systematic way to tell when a lower projective must be subtracted from the envelope.

One illuminating way to try to do this is to note that we can work in any finite $H_n^N(q)$ large enough to contain all the possible weights. There, for each indecomposable projective P_λ there is a (not necessarily unique) primitive idempotent e_λ such that $P_\lambda \cong H_n^N e_\lambda$, and Ind $P_\lambda \cong H_{n+1}^N e_\lambda$ (e_λ is the image in H_n^N of a corresponding primitive idempotent in H_n). The question is if e_λ can be decomposed in H_{n+1}^N. Generically it can, and there are constructions for the components (see e.g. [32]), but these components need not be well defined in every specialization of q. For example, the unique primitive idempotent $e_1 = Y_1 \in H_1^N$ is 1. In $H_2^N (N > 1, l \neq 2)$ this decomposes as

$$e_1 = 1 = \frac{q^{-1} - g_1}{[2]} + \frac{q + g_1}{[2]} = \begin{cases} e_0 + e_2 & N = 2 \\ e_{(1^2)} + e_{(2)} & N > 2 \end{cases} \tag{15}$$

(we have taken $g_i = q^{-1} - U_i, [2] = q + q^{-1}$, and adopted the convention $e_{(1^N)} = Y_N$, cf [50]). Obviously this decomposition is not available if $[2] = 0$. In general, if the idempotent decomposes we say Ind P_λ *splits*. If it does not split the induced character is the character for the new (higher) indecomposable projective ($P_{\lambda+e_1}$, say); if it does split the new character is the induced character less the lower character(s) which split off. The 'degree of divergence' as the generic idempotent is specialized to a given q can, in principle, be computed (cf the 'big diamond' idea of [45], and see section 4), but luckily for $M = 0$ there is a quicker way! We will discuss this shortly.

Continuing to iterate as below equation (14) we may deduce envelopes for projectives with weights in translates of any of the Weyl group reflections of A^0. Now suppose that all the projectives in a neighbourhood below ν *coincided* with their envelopes obtained in this way. It is straightforward to check that none of the projective envelopes we have generated contain complete copies of any of the lower ones. We may thus deduce that all our projectives coincide with their envelopes.

We will see that the supposition is correct for all but a few weights outside the 'forward light cone' of $\nu = (2l-1)\rho$ (cf [42]). Thus this procedure determines almost all projectives, leaving only those in a neighbourhood of the boundary of the dominant region open to question.

In the cases $N = 2, 3$ there is enough colateral information available to deduce the complete structure. We next discuss these cases. Then in section 5 we apply Soergel's recent work [58], which, in principle, determines all the projective subtractions for $M = 0$.

3.6. The case $N = 2$

The case $N = 2$ is the Temperley–Lieb algebra. This is well understood [46], and serves here to illuminate the notation. In this case the dominant region corresponds to the half-line, and dominant weights are the non-negative integers (corresponding, at fixed n, to the overhang of the top row of the Young diagram over the second row). Reflection hyperplanes are the points $l-1, 2l-1, \ldots,$ and the alcoves are simply the segments of the line bounded by these points.

For every N (and $l > N$) we have

$$P'_0 = \Delta'_0. \tag{16}$$

Inducing equation (16) in the case $N = 2$ we get $P'_1 = \Delta'_1$, then

$$\chi(P'_2) \leqslant \chi(\mathrm{Ind}\, P'_1) = \chi(\Delta'_2) + \chi(\Delta'_0).$$

The idempotent which splits these sections in the generic case at $n = 2$ is still well defined at $l > 2$ (indeed it is as in equation (15)), so $P'_2 = \Delta'_2$. We may iterate similarly up the fundamental alcove until we reach $\chi(P'_l) \leqslant \chi(\Delta'_l) + \chi(\Delta'_{l-2})$. By a direct calculation of idempotents [46] these sections do not split, so the bound is saturated. By the same calculation $P'_{ml-1} = \Delta'_{ml-1} = L'_{ml-1}$ for positive m, so $\chi(P'_{ml-1+k}) \leqslant \chi(\Delta'_{ml-1+k}) + \chi(\Delta'_{ml-1-k})(k < 1)$, and again we find that the bound is saturated (for $m > 1$ this is by induction on m, since then Δ'_{ml-1-k} is not projective).

We have determined that the non-singleton blocks of $N = 2$ are indexed by the dominant weights in the fundamental alcove, and that each such block gives a direct summand $D(\lambda)$ of D (from equation (5)):

$$D(\lambda) = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ & & 1 & 1 & \\ & & & & \cdots \end{pmatrix} \qquad D(\lambda)(D(\lambda))' = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 0 & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & & & \cdots \end{pmatrix}$$

with the ith row/column corresponding to the weight obtained by a sequence of $(i-1)$ affine reflections starting from the defining weight λ.

3.7. The case $N = 3$

For the case $N = 3$ it is useful first to note that the corresponding $N = 2$ case appears as a quotient for each fixed n (i.e. on the n-horizon). Note in particular that the affine reflection lines perpendicular to this line intersect it at the $N = 2$ reflection points. This means that the standard module content of an indecomposable projective on the n-horizon must be consistent with its (indecomposable projective) image in the quotient. Standard modules on the n-horizon are taken to (identical) standard modules in the quotient [50].

Note also that duality tells us that the global algebra has a symmetry corresponding to the usual * involution on the $U_q(sl_N)$ side (cf [31, p 311]).

The rest of the argument is an induction on the dominance order, starting from a number of explicit base cases.

Starting again from equation (16) we get $P'_{(1)} = \Delta'_{(1)}$, then $P'_{(2)} \subseteq \mathrm{Ind}\, P'_{(1)} = \Delta'_{(2)} + \Delta'_{(1,1)}$. Again this splits by a direct calculation (of the $n = 2$ q-symmetrizer), so $P'_{(2)} = \Delta'_{(2)}$ and $P'_{(1,1)} = \Delta'_{(1,1)}$, and so on.

The first non-split cases are of the type

$$\chi(P'_{(l-1,1)}) = \chi(\Delta'_{(l-3)}) + \chi(\Delta'_{(l-1,1)}) \tag{17}$$

and

$$\chi(P'_{(l)}) = \chi(\Delta'_{(l)}) + \chi(\Delta'_{(l-1,1)}) \tag{18}$$

(the reader is invited to mark the appropriate weights in figure 3). These cases cannot split for the following reason. First, note by induction and linkage that the right-hand sides shown are upper bounds. It is well known that the standard module $\Delta'_{(l-3)}$ is not irreducible in general (this is of the essence of the restricted ABF models [6]) and in particular for $n = l$. Thus in this case the weight must be linked to *something*. But the only possibility (noting equation (18) as an upper bound) is $\Delta'_{(l-1,1)}$. This shows that equation (17) does not split, which verifies that equation (18) cannot split either, since there is no projective summand (this is also signalled by the readily computed divergence of the $n = l$ q-symmetrizer, or by inspecting $N = 2$ data on the relevant n-horizon).

Note that all (non-vanishing) indecomposable projectives with weights within the same facet have category equivalent standard sections. Suppose without loss of generality that the sections of P'_λ are $\{\Delta'_{w\lambda} | w \in S \subset W_l\}$. If $\mu \in F(\lambda)$ adjacent to λ it is always possible to choose I^μ_λ so that $I^\mu_\lambda \Delta'_\lambda = \Delta'_\mu$ (in which case $R^\lambda_\mu \Delta'_\mu = \Delta'_\lambda$) or $R^\mu_\lambda \Delta'_\lambda = \Delta'_\mu$ (and $I^\lambda_\mu \Delta'_\mu = \Delta'_\lambda$), whereupon $w \in W_l$ implies $I^\mu_\lambda \Delta'_{w\lambda} = \Delta'_{w\mu}$, so the sections of P'_μ are $\{\Delta'_{w\mu} | w \in S\}$.

The next facet to consider is $F_{(2l-2)}$. Inducing $P'_{(2l-3)}$ and applying linkage the only possible sections are $\Delta'_{(2l-2)}$ and $\Delta'_{(l-1,l-1)}$. From the $N = 2$ quotient we find

$$\chi(P'_{(2l-2)} = \chi(\Delta'_{(2l-2)}) + \chi(\Delta'_{(l-1,l-1)}). \tag{19}$$

Inducing again we have

$$(P'_{(2l-1)} : \Delta'_\mu)_{\Delta'} \leqslant (I^{(2l-1)}_{(2l-2)} P'_{(2l-2)} : \Delta'_\mu)_{\Delta'} = (\Delta'_{(2l-1)} : \Delta'_\mu)_{\Delta'} + (\Delta'_{(l-2,\,l-2)} : \Delta'_\mu)_{\Delta'} \tag{20}$$

$$\chi(P'_{(2l-1,1)}) \leqslant \chi(\Delta'_{(2l-1,1)}) + \chi(\Delta'_{(2l-3)}) + \chi(\Delta'_{(l-1,\,l-2)}) + \chi(\Delta'_{(l,\,l)}) \tag{21}$$

(the reader is invited to mark the relevant weights on figure 3). The first of these bounds is saturated, for, supposing $I^{(2l-1)}_{(2l-2)} P'_{(2l-2)}$ splits we would have $\Delta'_{(2l-1)}$ projective, but then $R^{(2l-2)}_{(2l-1)} \Delta'_{(2l-1)} = \Delta'_{(2l-2)}$ would be projective—a contradiction. The second bound is saturated since any possible splitting would remove $\Delta'_{(l-1,l-2)}$, giving a contradiction on restriction to $P'_{(2l-1)}$.

At this point it is worth noting how the calculations so far appear in the context of the complete result, as given in the form of figures 4 and 5. For example, the four terms on the right-hand side of equation (21) correspond to the pattern in the $stuA^0$ alcove in figure 4.

To conclude the calculations we need one more technical device for idempotent splitting.

4. On generalized characters and idempotent splitting

Let $\{\sigma_i = (i\ i+1) | i = 1, 2, \ldots, n-1\}$ be the Coxeter generators of S_n as before. Recall that the *length* of a permutation w, written $\mathrm{len}(w)$, is the number of generators in a reduced expression for w. Let w_0 be the unique longest w in S_n. Let $T_w \in H_n$ be obtained by writing w reduced and replacing each σ_i by $g_i = q^{-1} - U_i$ (note that this procedure is well defined).

With $g_i = q^{-1} - U_i$ (note a simple change of variables cf [50]) the q-Young operators are given by

$$Y_n = \frac{1}{[n]!} \sum_{w \in S_n} q^{\mathrm{len}\,(w_0) - \mathrm{len}\,(w)} T_w \tag{22}$$

$$X_n = \frac{-1}{[n]!} \sum_{w \in S_n} (-q^{-1})^{\mathrm{len}\,(w_0) - \mathrm{len}\,(w)} T_w \tag{23}$$

where $[n] = q^{n-1} + q^{n-3} + \cdots + q^{1-n} = \frac{q^n - q^{-n}}{q - q^{-1}}$. Obviously $\{T_w | w \in S_n\}$ is a basis of H_n so these operators are not well defined in specializations of q in which $[l] = 0$ for some $l \leqslant n$ (i.e. q a $2l$th root of unity in these variables).

However, the operators are not necessarily badly defined in specializations of quotient algebras, in the sense that badly defined parts be killed by the quotienting. For example, writing $Y_2 = 1 - \frac{U_1}{[2]}$ we have

$$\begin{array}{c} 0 \rightarrow H_2 U_1 H_2 \rightarrow H_2 \rightarrow H_2^1 \rightarrow 0 \\ Y_2 \mapsto 1 \end{array} \tag{24}$$

so Y_2 is well defined in H_2^1, even when $[2] = 0$.

This observation leads (at least for projective modules) to a notion of generalized characters $\chi^v(M)$, with entries in $\mathbb{Z}[v]$, such that $\chi^1(M) = \chi(M)$. (We know of no direct physical use for the extra information, but knowing the generalized character of an induced module helps us extract the indecomposable projective characters). Let e_λ be a generic primitive idempotent as before (note that e_λ is not specified uniquely by λ in general). If it is well defined at l then $\chi^v_\mu(P_\lambda) = \delta_{\mu\lambda}$. If it is divergent we can characterize its divergence as follows. First, we return to the generic setting (indeed, go to $q = 1$), and examine the spin chain representation. Write

$$V_N^{\otimes n} = \bigoplus_\mu C_\mu \tag{25}$$

for the decomposition of $V_N^{\otimes n}$ by 'N-colour charge' conservation in the usual way. Obviously the permutation module C_μ is a direct sum of tilting modules, and thus has a Δ'-filtration. There are quotients H_n^ρ of H_n^N which will kill all C_μs with $\mu \not\trianglerighteq \rho$. The matrix elements of $C_\mu(e_\lambda)$ are rational numbers which may or may not be well defined in characteristic l (with an equivalent 'quantum' statement).

In the case of $e_{(n)}$ (which *is* uniquely defined) we can easily be completely specific. For example, let D_d be the $d \times d$ matrix with all entries 1, then

$$C_{(5,0,0)}(e_{(5)}) = \frac{1}{5!} \sum_{w \in S_5} C_{(5,0,0)}(w) = \frac{1}{5!} D_1 = 1 \tag{26}$$

$$C_{(4,1,0)}(e_{(5)}) = \frac{1}{5!} \left\{ \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix} + \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0 \end{pmatrix} + \cdots \right\} = \frac{4!}{5!} D_5 \tag{27}$$

$$C_{(3,2,0)}(e_{(5)}) = \frac{3!2!}{5!} D_{\frac{5!}{3!2!}} \tag{28}$$

$$C_{(3,1,1)}(e_{(5)}) = \frac{3!}{5!} D_{\frac{5!}{3!}} \tag{29}$$

$$C_{(2,2,1)}(e_{(5)}) = \frac{2!2!}{5!} D_{\frac{5!}{2!2!}}. \tag{30}$$

Put $d_\mu = \dim(C_\mu) = \frac{n!}{\Pi_i \mu_i!}$. The general result is

$$C_\mu(e_{(n)}) = \frac{1}{d_\mu} D_{d_\mu}. \tag{31}$$

These results follow from elementary representation theory considerations. The situation is considerably more complicated for general e_λ, but we will not need it here.

In our $n = 5$ case the dominance order totally orders the weights. We see that $e_{(5)}$ is well defined in $H_5^{(5,0,0)} = H_5^1$ for any l. In $H_5^{(4,1,0)}$ it is well defined unless $l = 5$ (note that the denominator cannot be removed by base change, since, more generally, the HCF of a matrix's elements is invariant under unimodular transformation). We deduce that $e_{(5)}$ is not well defined at $l = 5$, and that the nature of the divergence is such that some choice of idempotent $e_{(4,1)}$ must be added to make an idempotent which *can* be specialized to $l = 5$ (we knew this already, but now we have a well defined degree of divergence, i.e. the power of 5 in the denominator when equation (27) is expressed in reduced form). Note that since the degree of divergence at $l = 5$ is the same in each subsequent quotient, then if further idempotents needed to be added (in this case they do not) their degree would be no higher than 1. In $H_5^{(3,2,0)} = H_5^2$ the idempotent $e_{(5)}$ is then well defined unless $l = 4$, so for $l = 4$ some $e_{(3,2)}$ must be added, and so on.

Now recall that $\chi(P_\lambda)$ records which e_μs must be added to e_λ to make a well defined idempotent at l. In $\chi^v(P_\lambda)$ we simply record the degree of divergence of each addition, i.e. (in our case) the power of l in the denominator when equation (31) is expressed in reduced form. With the machinery we have described so far, we can note the largest $\mu(s)$ where $1/l$ first appears, the largest where $1/l^2$ appears, and so on (care must be taken when interpreting for the quantum case when l not prime—obviously [2][2] does not have all the roots of [4], for example).

Note that for $N = 2, 3$ the polynomials $\chi^v(P_\lambda)$ are all monomials with coefficient 1 (or 0). An illuminating example occurs at $\lambda = (12)$. Here we have $d^{-1}_{(12)} = 1$, $d^{-1}_{(11,1)} = \frac{1}{12}$, $d^{-1}_{(10,2)} = 2/(12.11), \ldots, d^{-1}_{(7,3,2)} = 7!3!2!/12!, \ldots, d^{-1}_{(4,4,4)} = 4!4!4!/12!$. Considering $l = 4$ (figure 8) we pick out $(11, 1)$ at degree 1, $(7, 3, 2)$ at degree 2. Noting $(8, 4)$ at degree 0 and $(8, 3, 1)$ at degree 1 we arrive at the result in figure 4, where the appropriate subset of these degrees are shown in the pattern for $\lambda = (12)$ given in the key on the right. Indeed, with a couple of similar examples this concludes the base of the $N = 3$ calculation, which then concludes as in section 3.5. Furthermore, we can infer, in general, that the degrees of divergence do not grow unboundedly with n, as naively suggested by equation (22) or (23), but are rather limited by N.

5. General N results

We want to associate to every facet of $\mathcal{A}^+_-$, or more precisely to a representative weight λ in every facet (and hence via equivalence translations T^μ_λ—see equation (34)—to every simple module index) a map encoding the standard module content of the corresponding U_q tilting module, or equivalently the standard content of each H^N_n projective. By the reciprocity of equation (6) this will also lead us to the simple content of the standard Hecke algebra modules. We will do this explicitly for alcoves and walls, with the other facets following by suitable induction/restriction (or translation—see appendix A). We work iteratively on the Bruhat order for $\mathcal{A}^+$ and $\mathcal{A}^+_1$.

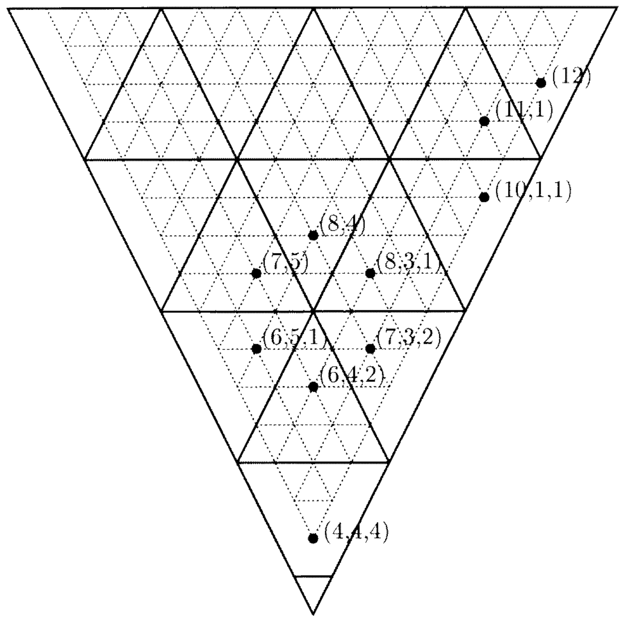

Figure 8. Some A_2 weights in the 4-affine Weyl group orbit of $\lambda = (12)$. Note the $N = 3$ equivalences $(8, 3, 1) \sim (7, 2)$ and so on.

For each $A \in \mathcal{A}^+$ pick a Bruhat increasing path to it from A^0 via the right action (such a path is not unique, and in fact we only choose one for definiteness). We now compute for each $A \in \mathcal{A}$ a map

$$n_A : \mathcal{A} \to \mathbb{Z}[v]$$

and for each $w \in \mathcal{A}_1^+$

$$n_w : \mathcal{A}_1^+ \to \mathbb{Z}[v]$$

such that

$$n_A(A) = 1 \text{ and } n_A(B) \neq 0 \text{ implies } B = A \text{ or } B < A \tag{32}$$

(note that n_{A^0} is defined uniquely by this).

We start with n_{A^0} and work up the order as follows. For $As > A$

$$n_{w_s(A)} : \mathcal{A}_1^+ \to \mathbb{Z}[v]$$

is given by

$$n'_{w_s(A)}(w_s(B)) = \begin{cases} n_A(B) + v^{-1} n_A(Bs) & Bs > B \\ v^{-1} n_A(B) + n_A(Bs) & Bs < B \end{cases}$$
$$n'_{w_s(a)}(w) = 0 \qquad w \notin [s]$$

(note that this is not obviously well defined, but in fact every $n_A(B)$ not divisible by v is 'projected up' by w_s); and then finally

$$n_{w_s(A)} = n'_{w_s(A)} - \sum_{w < w_s(A)} n'_{w_s(A)}(w)|_{v=0} n_w.$$

Now given

$$n_{w_s(A)} : \mathcal{A}_1^+ \to \mathbb{Z}[v]$$

with $As > A$ put

$$n_{As}(B) = \begin{cases} n_{w_s(A)}(w_s(B)) & B > Bs \\ v n_{w_s(A)}(w_s(B)) & B < Bs. \end{cases}$$

Note that these polynomials are well defined, independent of the path (by a braiding argument) and crucially, the key result is that

$$(T_\lambda : \Delta_\mu) = n_{F_\lambda}(F_\mu)|_{v=1}$$

whenever the right-hand side is well defined (Soergel [58] proves the $\mathcal{A}^+$ part of this result; see also [9, 25, 38]; the remainder follows from properties of the translation functor [34]).

The cases of Coxeter systems A_1, A_2 should be compared with sections 3.6 and 3.7 (see also Lusztig [42]). In fact, the A_2 example is again illustrated by figures 4 and 5. The figures are now to be interpreted as follows. There are three types of facet: faces, edges and points. The procedure above only gives the face and edge data, but recall that all the tilting modules corresponding to *points* on the alcove diagram here *coincide* with the standard modules with the same label (i.e. $n_A(A) = 1$ and all other $n_A(B) = 0$; see e.g. Jantzen [34, ch 8]. Thus we need only encode the set of polynomials $n_A(-)$ for each class A of edge and face. Fixing A, there is a polynomial for each $n_A(B)$, B an edge or face, respectively. In each case of A it turns out that almost all the polynomials are zero (of course this if forced by equation (32), but the number of non-zero entries is actually much smaller than this constraint requires). If we look at the pattern of non-zero polynomials for given A we find that (up to translation of the pattern bodily around the picture) there are only a small number of distinct patterns. Indeed, up to this translation even the details of the non-zero polynomials are fixed in a given pattern. Thus we can give all the polynomials by simply describing which generic pattern type is associated to each A. Strictly speaking we also need to know the bodily position of the given pattern in the picture, but that is fixed by the position of the 'head' of the pattern, the polynomial $n_A(A) = 1$ (by equation (32) the unique highest alcove in the pattern), at position A. Note that there is only one polynomial '1' in each pattern.

In the figures, the generic pattern types are shown by the templates on the right. Each non-zero polynomial is of the form $n_A(B) = v^x$ (this is generally true for $N = 2, 3$ only); and x is given by the number in the template. Thus, for example, the head of each pattern is marked 0, for v^0. The shapes in the alcove diagram itself indicate which template applies for each alcove/wall (if only part of a generic pattern is shown then the generic polynomials are replaced by zero in the omitted alcoves). For example, from figure 4, $n_{sA^0}(A^0) = v$.

Results for higher N are easy to compute but, due to the higher-dimensional alcove diagrams, harder to present (consider figure 2!). We will deal with the analysis of these results elsewhere.

To finish, let us reiterate in short. Even without using the full polynomial data, figures 4 and 5 yield all composition multiplicities for $U_q(sl_3)$, and hence all corresponding spectrum multiplicities for the Hamiltonians $\mathcal{H}^{3,0}$. Two complementary methods have been given for determining these. The method for determining the multiplicities for arbitrary N is also given. We note for completeness that the full polynomial data gives information about the position of filtration factors in the corresponding filtrations (cf Jantzen [34, ch 8; [4, 49]), i.e. where appropriate, the Loewy layer [50]. However, this extra data does not of itself seem to be physically interesting.

While Soergel's approach [58] to the procedure discussed here uses and is restricted to algebraic Lie theory, the method in sections 3 and 4 does not have this restriction in principle (see [49]). In particular, it should be possible to generalize to the algebras of the reflection equation [8]. We will look at this problem elsewhere.

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Appendix. Representation theory generalities

To introduce tilting modules [28] we can proceed either via q-groups [13] or *quasi-hereditary* algebras [22]. In the diagram in equation (4) $U_q(sl_N)$ is the q-group (a Hopf algebra) while the finite-dimensional quotients $S_q(n, N)$ are quasi-hereditary. Since the construction of the q-group is relatively complicated we will here follow the way of quasi-heredity. Recall, however, that the name q-group derives justification partly from the appearance of q-groups as the 'symmetry groups' of n-site q-spin chains [54] in the sense that chains are invariant under an action of the q-group. For any given n this is not a faithful action, i.e. the invariance is fully realized by the action of some *finite* quotient algebra, which is not itself a Hopf algebra. These quotients $S_q(n, N)$ have nice properties both inherited from the q-group and due to finiteness [27]. We study these quotients.

An idempotent e in a finite-dimensional algebra A over $\mathbb{C}$ is a *heredity idempotent* if eAe is semi-simple and the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection. A heredity chain for A is a list $(e_0, e_1, \ldots, e_m)$ of idempotents such that $A = Ae_0A \supset Ae_1A \supset \cdots \supset Ae_mA$ and e_i is a heredity idempotent modulo $Ae_{i+1}A$. An algebra with heredity chain is called quasi-hereditary [22]. Any heredity chain can be refined so that each e_i has a primitive image in $A_i := A/Ae_{i+1}A$, whereupon the chain is said to be maximal (of length $m_{\max}$, say).

Let A be quasi-hereditary with maximal heredity chain. Then define *standard* modules by restriction along the natural projection from A to A_i

$$\Delta_i := \mathrm{Res}_A^{A_i}(A_i e_i).$$

Let the set of these be Δ. Dually there are *costandard* modules $\nabla_i \in \nabla$ given by $\nabla_i = (\mathrm{Res}_A^{A_i}(e_i A_i))^*$. The heads $L_i = \mathrm{Head}(\Delta_i)$ are a complete set of inequivalent simple A modules. The set $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is the set of *tilting modules* of A.

Now let X' be a fixed index set of inequivalent simple modules (so $X' \leftrightarrow \{1, 2, \ldots, m_{\max}\}$). Let $>$ be the partial order on X' defined by

$$(\lambda > \mu) \Rightarrow (L_i \cong L_\lambda, L_j \cong L_\mu \Rightarrow i > j \text{ for every maximal chain of } A).$$

It is a theorem of Ringel [57] that the set of equivalence classes of indecomposable tilting modules may be indexed by X', and that representative T_λ (say) may be characterized by

$(T_\lambda : \Delta_\lambda) = 1$ and $(T_\lambda : \Delta_\mu) = 0$ for $\mu \not\leq \lambda$. A 'full' tilting module is one containing a direct sum of at least one of each $T_\lambda (\lambda \in X')$.

Using the ideas of Donkin [28] one can also define tilting modules for q-groups, with the role of Δ played by the set of Weyl modules. In this setting one can, in principle, give a construction of the indecomposable tilting modules via tensor products, using Donkin's result that T_a and T_b tilting implies $T_a \otimes T_b$ tilting. For example, $V_N^{\otimes n}$ is tilting for $U_q(sl_N)$, since V_N is trivially so. The quotient algebra $S_q(n, N)$ is quasi-hereditary [62], and $V_N^{\otimes n}$ is tilting as a $S_q(n, N)$-module. By an organizational argument $V_N^{\otimes n}$ is a full tilting module in $S_q(n, N)$ provided $l > N$.

If A is a quasi-hereditary algebra and T a (left) full tilting module then the algebra $A' := \mathrm{End}_A(T)$ is called the Ringel dual of A with respect to T. It can be shown that this dual is quasi-hereditary. Note that T is a left A right A'-module, so that the functor $F = \mathrm{Hom}_A(T, -)$ takes

$$F : A\text{-}\mathrm{mod} \to A'\text{-}\mathrm{mod}\,.$$

In particular, F takes injective modules to tilting modules. Consider A itself as a right A-module, then

$$A_A^* = \bigoplus_\lambda {}_A I(\lambda)$$

is a sum of left injective modules. Thus $T' := F(A_A^*)$ is full tilting. We have

$$T' = \mathrm{Hom}_A(T, A_a^*) \cong \mathrm{Hom}_A(A, T^*) = T^*$$

an isomorphism of left A'-modules, thus finally A' has (right) full tilting module T. The heredity order $>$ is reversed in A'.

From the above discussion $H_n^N(q)$ is quasi-hereditary (provided $l > N$), being the Ringel dual of $S_q(n, N)$ with respect to $V_N^{\otimes n}$. The connection between the algebras is provided by the Ringel functor

$$\begin{aligned} &F : S_q(n, N)\text{-mod} \to H_n^N(q)\text{-mod} \\ &F(M) = \mathrm{Hom}_{S_q(n,N)}(V_N^{\otimes n}, M). \end{aligned} \tag{33}$$

This takes costandards to standards and preserves exactness of Δ-filtered sequences. In particular, $\{P'_\lambda = F(T_\lambda) | \lambda \in X^n\}$ defines a complete set of inequivalent indecomposable projective modules of $H_n^N(q)$. This labelling of the modules coincides with that in [50].

Let $\mathrm{Pr}_\lambda : U\text{-mod} \to \mathcal{M}_\lambda$ be the projection functor corresponding to equation (7). Then for $\lambda, \mu \in C$ the *translation functor* [34] $T_\lambda^\mu : U\text{-mod} \to \mathcal{M}_\mu$ is defined by

$$T_\lambda^\mu M = \mathrm{Pr}_\mu (L((\mu - \lambda)^+) \otimes \mathrm{Pr}_\lambda M) \tag{34}$$

where $(\mu - \lambda)^+$ is the dominant conjugate of $\mu - \lambda$. Translation involving indices μ, λ in the same facet is a category equivalent [34], so in examining the structure of standard modules we may restrict attention to one representative weight in each facet, with all the other data being straightforwardly recoverable from these.

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